# Annex: Sampling uncertainty in the ISO 3951-1 standard

In the following, the manner in which sampling uncertainty is taken into consideration in the ISO 3951-1 standard will be explained.

### “s” method

Assume that a lower specification limit $L$ has been defined and that a sample of size $n$ has been drawn. The approach in the ISO 3951-1 standard consists in defining an *acceptability constant* $k$ in such a way that the criterion for accepting the lot or consignment, from which the sample was drawn, is:

$$\overbar{X}-kS\geq L,$$

where $\overbar{X}$ denotes the sample mean and $S$ denotes the sample standard deviation.

This is where sampling uncertainty comes in. Due to the random nature of the sample, it cannot be excluded that the sample misrepresents the lot or consignment, thus leading to an incorrect decision. Indeed, the computation of $k$ does not depend on the measured values obtained on the basis of the sample. Thus, once $k$ has been computed, different samples will yield different values for the quantity $\overbar{X}-kS$. Taking sampling uncertainty into consideration will thus consist in a description of the distribution of the values which the quantity $\overbar{X}-kS$ takes and, in particular, in the quantification of the probabilities of making incorrect decisions. There are two types of incorrect decisions:

* Accepting a consignment which does not meet the criterion (the consumer’s risk: $\overbar{X}-kS\geq L$ even though the true average of X lies below the specification limit L)
* Rejecting a consignment which meets the criterion (the producer’s risk: $\overbar{X}-kS<L$ even though the true average of X lies above the specification limit L)

In the ISO 3951-1, the starting point for taking sampling uncertainty into consideration thus consists in a decision as to which of the two types of risks to prioritize. In the “standard procedure” (see Section 13.1), the order of priority is:

1. Producer’s risk
2. Sample size
3. Consumer’s risk

(Procedures for other priority orders are also provided (see Section 13.1).)

It is assumed that the producer and the consumer have each defined a percentage of items nonconforming on the basis of which to establish the acceptance criterion:

* The consumer’s percentage of items nonconforming is called the “Limiting Quality”
* The producer’s percentage of items nonconforming is called the “Acceptance Quality Limit” (AQL)

(Once the Limiting Quality has been defined by the consumer, the producer can choose the AQL to be lower than the Limiting Quality so as to ensure a high probability of acceptance.)

If the producer’s risk is prioritized, the acceptability constant $k$ will be computed in such a way so as to ensure the producer’s risk is low (say, less than $5 \%$). Once this particular $k$ has been computed, it is possible to compute the consumer’s risk. Typically, the latter is deemed acceptable if less than $10 \%$. If, for the computed $k$, the consumer’s risk is too high, then it is possible to lower this risk by increasing the number of samples. The sample sizes corresponding to the code letters in the tables of the ISO 3951-1 standard were computed to ensure the criteria for both the producer risk (less than $5 \%$) and the consumer risk (less than $10 \%$) are met. In particular, tables B1, K1 and L1 provide the information discussed here:

* Table B1 provides the $k$ values (“acceptability constant”) corresponding to a particular AQL and sample size.
* Table K1 provides the corresponding consumer risks.
* Table L1 provides the corresponding producer risks.

As far as the actual computations are considered, the non-central t distribution plays an important role as can be read from the following equation:

$$P\left(\overbar{X}-kS\geq L\right)=P\left(\frac{\sqrt{n}\left(\overbar{X}-μ\right)}{σ}+\frac{\sqrt{n}\left(μ-L\right)}{σ}\geq k\sqrt{n}\frac{S}{σ}\right),$$

$$=P\left(T\_{n-1, δ}\geq k\sqrt{n}\right),$$

where $T\_{n-1, δ}$ denotes a random variable following a non-central t distribution with non-centrality parameter $δ$.

In the computation of the acceptability constant $k$, the producer’s risk is prioritized. The percentage of items nonconforming $AQL=P(X<L)$ can be written $AQL=P\left(\frac{X-μ}{σ}<\frac{L-μ}{σ}\right)=Φ\left(\frac{L-μ}{σ}\right)$.

For the non-centrality parameter, this implies

$$δ=\frac{\sqrt{n}\left(μ-L\right)}{σ}=-\sqrt{n}∙Φ^{-1}\left(AQL\right).$$

Finally, we have for the probability of acceptance:

$$P\_{a}=95 \%=1-F\_{n-1,δ}\left(k\sqrt{n}\right),$$

Where $F\_{n-1,δ}$ denotes the non-central t distribution function. The last equation can then be solved for $k$.

It can thus be seen that the computation of $k$ depends only on the choice of the $AQL$, on the sample size $n$ and on the producer’s risk.

The choice of the producer’s risk (e.g. 5 %) is the direct manifestation of sampling uncertainty (on the assumption that analytical uncertainty is negligible). If analytical uncertainty is not negligible, then the producer’s risk is caused by both sampling and analytical uncertainty. It should also be noted that the acceptance rule $\overbar{X}-kS\geq L$ depends indirectly on the sample standard uncertainty $u(\overbar{X})=S/\sqrt{n}$.

### “$σ$” method

The dependence of the decision rule on the sample standard uncertainty $u(\overbar{X})$ is more obvious for the “$σ$” method. For this method, the empirical standard deviation “S” is replaced by the theoretical (known) standard deviation “$σ$”.

The probability for acceptance for the “$σ$” method can be computed

$$P\left(\overbar{X}-kσ\geq L\right)=P\left(\frac{\sqrt{n}\left(\overbar{X}-μ\right)}{σ}+\frac{\sqrt{n}\left(μ-L\right)}{σ}\geq k\sqrt{n}\right),$$

$$=1-Φ\left(k\sqrt{n}-\frac{\sqrt{n}\left(μ-L\right)}{σ}\right),$$

In the computation of the acceptability constant $k$, again the producer’s risk is prioritized. Substituting the expression for the percentage of items nonconforming $AQL=Φ\left(\frac{L-μ}{σ}\right)$ (as above), we obtain $\frac{\sqrt{n}\left(μ-L\right)}{σ}=-\sqrt{n}∙Φ^{-1}\left(AQL\right).$

For the probability of acceptance we obtain:

$$P\_{a}=95 \%=P\left(\overbar{X}-kσ\geq L\right)=1-Φ\left(k\sqrt{n}+\sqrt{n}∙Φ^{-1}\left(AQL\right)\right).$$

The last equation can then be solved for $k$.

$$k\sqrt{n}+\sqrt{n}∙Φ^{-1}\left(AQL\right)=Φ^{-1}\left(1-P\_{a}\right).$$

$$k=Φ^{-1}\left(1-P\_{a}\right)/\sqrt{n}-Φ^{-1}\left(AQL\right).$$

As for the “s” method, $k$ thus depends only on $P\_{a}$, $AQL$ and on the sample size $n$. Again, the sampling uncertainty (in the absence of analytical uncertainty) is the sole source of random variability – and thus the producer’s risk is the acknowledged and controlled manifestation of the sampling uncertainty.

#### An alternative decision rule A for acceptance based on sample uncertainty and producer’s risk

With the following parameters:

* the sample average $\overbar{X}$ ,
* the sample uncertainty $u\left(\overbar{X}\right)=σ/\sqrt{n}$ ,
* the expanded uncertainty $U\_{A}\left(\overbar{X}\right)=Φ^{-1}\left(P\_{a}\right)∙σ/\sqrt{n}$ with the coverage factor $Φ^{-1}\left(P\_{a}\right)$ based on the producer’s risk (say, $P\_{a}=0.95=1-producer's risk$),
* the $AQL$ and
* the AQL process mean of the measured characteristic $μ\_{AQL}=L-σ∙Φ^{-1}\left(AQL\right)$ at which the fraction of nonconforming items equals AQL (by this definition of $μ\_{AQL}$ we have $AQL=Φ\left(\frac{L-μ\_{AQL}}{σ}\right)$),

the decision rule A is defined as follows: The lot will be rejected if the uncertainty interval $\overbar{X}\pm U\_{A}\left(\overbar{X}\right)$ is completely below the critical process mean $μ\_{AQL}$ without overlap, i.e. if $\overbar{X}$ is significantly higher than $μ\_{AQL}$, in other words, if the expected rate of non-conforming items is significantly higher than AQL. The lot will be accepted if $\overbar{X}+U\_{A}\left(\overbar{X}\right)>μ\_{AQL}$.

It can be shown easily that this decision rule A is equivalent to the ISO 3951-1 approach described above.

Indeed, the criterion

$$\overbar{X}+U\_{A}\left(\overbar{X}\right)>μ\_{AQL}$$

is equivalent to

$$\overbar{X}+Φ^{-1}\left(P\_{a}\right)∙σ/\sqrt{n}>L-σ∙Φ^{-1}\left(AQL\right)$$

and

$$\overbar{X}-σ∙\left(\frac{Φ^{-1}\left(1-P\_{a}\right)}{\sqrt{n}}-Φ^{-1}\left(AQL\right)\right)>L.$$

The expression in the brackets equals k, i.e. the decision rule for acceptance is equivalent with $\overbar{X}-σk>L, $and this is the approach in ISO 3951-1.

#### Another decision rule B for acceptance based on sample uncertainty and consumer’s risk

Another decision rule B based on consumer’s risk can be defined with the following parameters:

* the expanded uncertainty $U\_{B}\left(\overbar{X}\right)=Φ^{-1}\left(0.9\right)∙σ/\sqrt{n}$ with the coverage factor $Φ^{-1}\left(0.9\right)$ based on the consumer’s risk,
* the LQ process mean $μ\_{LQ}=L-σ∙Φ^{-1}\left(LQ\right)$ at which the fraction of nonconforming items equals the limiting quality LQ.

The lot will be accepted if the uncertainty interval $\overbar{X}\pm U\_{B}\left(\overbar{X}\right)$ is completely above the LQ process mean $μ\_{LQ}$ without overlap, i.e. the lot will be rejected if $\overbar{X}-U\_{B}\left(\overbar{X}\right)<μ\_{LQ}$.

LQ is defined in such a way that decision rule B is equivalent with decision rule A.

#### Example

Consider the sampling scheme described in Table C.1 (ISO 3951-1) for code letter H with n=12 samples. Let AQL = 1% the acceptance quality limit, $σ=2$ the standard deviation, and L = 20 the lower specification limit. Then k = 1.800 according to Table C.1 and LQ = 7.64% according to Table K.2 (ISO 3951-1).

With these parameters the AQL process mean $μ\_{AQL} $and the LQ process mean $μ\_{LQ}$ can be calculated as follows:

|  |  |
| --- | --- |
| **Decision rule A: based on producer’s risk** | **Decision rule B: based on consumer’s risk** |
| AQL=1%, $σ=2$, L = 20 | LQ=7.64%, $σ=2$, L = 20 |
| $$Φ^{-1}\left(AQL\right)=-2.326$$ | $$Φ^{-1}\left(LQ\right)=-1.430$$ |
| $$μ\_{AQL}=L-σ∙Φ^{-1}\left(AQL\right)$$ | $$μ\_{LQ}=L-σ∙Φ^{-1}\left(LQ\right)$$ |
| $μ\_{AQL}=24.65$ **(red line)** | $μ\_{LQ}=22.86$ **(red line)** |
| AQL | LQ |

In order to illustrate the decision rules, we assume the sample average over the 12 samples $\overbar{X}=23.8$.

The uncertainty intervals and the final decision-making is as follows:

|  |  |
| --- | --- |
| **Decision rule A: based on producer’s risk** | **Decision rule B: based on consumer’s risk** |
| Producer’s risk = 0.05 | Consumer’s risk = 0.10 |
| Coverage factor: $Φ^{-1}\left(1-0.05\right)=1.645$ | Coverage factor: $Φ^{-1}\left(1-0.10\right)=1.282$ |
| $$U\_{A}\left(\overbar{X}\right)=Φ^{-1}\left(1-0.05\right)∙\frac{σ}{\sqrt{n}}=0.95$$ | $$U\_{B}\left(\overbar{X}\right)=Φ^{-1}\left(1-0.10\right)∙\frac{σ}{\sqrt{n}}=0.74$$ |
| Uncertainty interval: $\overbar{X}\pm U\_{A}\left(\overbar{X}\right)=23.8\pm 0.95$ (represented by green horizontal bar) | Uncertainty interval: $$\overbar{X}\pm U\_{B}\left(\overbar{X}\right)=23.8\pm 0.74$$(represented by green horizontal bar) |
| Decision:$$\overbar{X}+U\_{A}\left(\overbar{X}\right)=24.75>μ\_{AQL}=24.65$$Lot accepted (green bar not completely left from red line; still overlapping)  | Decision:$$\overbar{X}-U\_{B}\left(\overbar{X}\right)=23.06>μ\_{LQ}=22.86$$Lot accepted(green bar right from red line; no overlap)  |
|  |  |

It should be noted that both decision rules are close at the limit. This is true also for the acceptance criterion of the ISO approach,$ \overbar{X}-kσ>L$, as

$$\overbar{X}-kσ=23.8-1.8∙2=20.2>L=20.$$

If $\overbar{X}$<23.6, the three decision rules will come to a negative result, ie the lot will be rejected. This example demonstrates that the three decision rules are indeed equivalent, ie whether the lot will be accepted or not is not depending on the choice of the rule.

## Conclusion

1. The decision as to the acceptance of a lot according to the ISO approach for the $"σ$” method is equivalent with decision rule A. Decision rule A is based on a comparison of the sample average with the AQL process mean under consideration of the sampling uncertainty. The coverage factor is derived from the corresponding producer’s risk. Decision rule A is in line with the GUM. Sampling uncertainty is taken into account, and analytical uncertainty is assumed to be negligible.
2. The decision as to the acceptance of a lot according to the ISO approach for the $"σ$” method is also equivalent with decision rule B. Decision rule B is based on a comparison of the sample average with the LQ process mean under consideration of the sampling uncertainty. The coverage factor is derived from the corresponding consumer’s risk. Decision rule B is in line with the GUM. Sampling uncertainty is taken into account, and analytical uncertainty is assumed to be negligible.
3. The ISO approach for the $"σ$” method could easily be replaced by one of the decision rules A or B. Both rules take into account sampling uncertainty.
4. Sampling uncertainty for the process mean µ according ISO 3951-1 is computed $(\overbar{X})=σ/\sqrt{n}$ , where $σ$ denotes the true standard deviation of the variability between items.
5. Also the ISO approach for the $"s$” method takes into account sampling uncertainty. Coverage is derived from both producer’s and consumer’s risk.
6. The toolbox provided by ISO 3951 is very powerful and applicable for many different situations. It is statistically valid and can be applied also in situations
	1. when the uncertainty is very large (a situation which is critical for the UfS approach)
	2. when the analytical uncertainty is not negligible (since the last revision of the standard).
7. On the other hand, the ISO approach uses complicated terminology, fails to provide proper explanation of statistical procedures and uses complicated tables and charts for decision-making. It is therefore suggested to
	1. develop more appropriate terminology
	2. produce explanations for statistical concepts
	3. simplify outdated procedures of decision-making.